Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 23 Issue 8 Version 1.0 Year 2023
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Anisotropic Conservative Caginalp Phase-Field System based on Type III Heat Conduction with Two Temperatures and Periodic Boundary Conditions

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GJSFR-F Classification: LCC Code: QA801-939

ONANISOTROPICCONSERVATIVECAGINALPPHASEFIELDSYSTEMBASEDONTYPEIIIHEATCONDUCTIONWITHTWOTEMPERATURESANDPERIODICBOUNDARYCONDITIONS

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[13] A. Miranville and R. Quintanilla, *A Caginalp phase-field system based on type III heat conduction with two temperatures*, Quart. Appl. Math., 2016, 74, 375–398.

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Keywords: conserved phase-field system, anisotropy, type III thermomechanics, two temperatures, well-posedness.

I. Introduction

The authors studied in [13] (see aigain [12]) the following phase-field system, namely,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \tag{1.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{1.2}$$

$$\alpha(t,x) = \alpha(0,x) + \int_0^t T(\tau,x)d\tau, \tag{1.3}$$

where, u is the order parameter, T is the relative temperature (defined as $T = \widetilde{T} - T_E$, where $\widetilde{T}$ is the absolute temperature and T_E is the equilibrium melting temperature), α is the conductive thermal displacement and f is the derivative of a double-well potential F (a typical choice is $F(s) = \frac{1}{4}(s^2-1)^2$, hence the usual cubic nonlinear term $f(s) = s^3 - s$). Furthermore, here and below, we set all physical parameters equal to one. This system has been introduced to model phase transition phenomena, such as melting-solidication phenomena, and has been much studied from a mathematical point of view. We refer the reader to, e.g., [4–5, 8–11, 14, 16, 17, 21, 23].

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This system is based on the (total Ginzburg-Landau) free energy,

$$\Psi_{\mathrm{GL}} = \int_{\Omega} (\frac{1}{2}|\nabla u|^2 + F(u) - uT - \frac{1}{2}T^2)\mathrm{d}x, \tag{1.4}$$

where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^n$, $n = 2$ or $n = 3$, with boundary Γ), and the enthalpy

$$H = u + T - \Delta T. \tag{1.5}$$

As far as the evolution equation for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial u} = -\frac{D\Psi_{GL}}{Du}, \tag{1.6}$$

where $\frac{D}{Du}$ denotes a variational derivative with respect to u. Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\mathrm{divq} \tag{1.7}$$

and owing to (1.7),

$$\frac{\partial T}{\partial t} - \Delta\frac{\partial T}{\partial t} + \mathrm{divq} = -\frac{\partial u}{\partial t}, \tag{1.8}$$

where q is the heat flux. Assuming finally the usual Fourier law for heat conduction,

$$q = -\nabla\alpha - \nabla T, \tag{1.9}$$

we obtain (1.1) and (1.2).

Our aim in this paper is to study the model consisting of the conserved anisotropic to (1.1)–(1.2), namely,

$$\frac{\partial u}{\partial t} + \Delta\sum_{i=1}^{3} a_i\frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta\left(\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right),\ a_i > 0, \tag{1.10}$$

$$\frac{\partial^2\alpha}{\partial t^2} - \Delta\frac{\partial^2\alpha}{\partial t^2} - \Delta\frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t}. \tag{1.11}$$

Our aim in this paper is to study the model consisting of the anisotropic conservative equation (1.10) and the temperature equation (1.11). In particular, we obtain the existence and uniqueness of solutions.

Notes

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II. Setting of the Problem

Notes

Find the order parameter u $:\Omega \times \mathbb{R}^+ \to \mathbb{R}$ and the termal displacement $\alpha : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ such that:

$$\frac{\partial u}{\partial t} + \Delta \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta \left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right), \tag{2.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{2.2}$$

together with periodic boundary conditions

$$u \quad \text{and} \quad \alpha \quad \text{are} \quad \Omega - \text{periodic}, \tag{2.3}$$

and the initial conditions

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{2.4}$$

We assume that

$$a_i > 0, \quad i \in \{1,2,3\}, \tag{2.5}$$

and we introduce the elliptic operator A defined by

$$\langle Av, w \rangle_{H^{-1}(\Omega), H^1_{per}(\Omega)} = \sum_{i=1}^{3} a_i \left(\left(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right) \right), \tag{2.6}$$

where $H^{-1}(\Omega)$ is the topological dual of $H^1_{per}(\Omega)$. Furthermore, $((.,.))$ denotes the usual L^2-scalar product, with associated norm $\|.\|$; more generally, we denote by $\|.\|_X$ the norm on the Banach space X and we set $\|.\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}.\|$, $(-\Delta)^{-1}$ denoting the inverse minus Laplace operator with periodic boundary conditions and acting on functions with null average, is a norm in $H^{-1}(\Omega) = H^1_{per}(\Omega)'$ which is equivalent to the usual H^{-1}-norm. We can note that

$$(v, w) \in H^1_{per}(\Omega)^2 \mapsto \sum_{i=1}^{3} a_i \left(\left(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right) \right)$$

is bilinear, symmetric, continuous and coercive, so that

$$A : H^1_{per}(\Omega) \to H^{-1}(\Omega)$$

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is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order 2 (see [1–2]) that A is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(A) = H^2_{per}(\Omega),$$

where, for $v \in D(A)$,

$$Av = -\sum_{i=1}^{3} a_i \frac{\partial^2 v}{\partial x_i^2}.$$

We further note that $D(A^{\frac{1}{2}}) = H^1_{per}(\Omega)$ and, for $v \in D(A^{\frac{1}{2}})$,

$$((A^{\frac{1}{2}}v, A^{\frac{1}{2}}v)) = \sum_{i=1}^{3} a_i \left\| \frac{\partial v}{\partial x_i} \right\|^2.$$

We finally note that (see, e.g., [18]) $v \mapsto (\|Av\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$ defines a norm on $H^2_{per}(\Omega)$ which is equivalent to the usual H^2-norm on $D(A)$ (resp., $v \mapsto (\|A^{\frac{1}{2}}v\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$ defines a norm on $H^1_{per}(\Omega)$ which is equivalent to the usual H^1-norm on $D(A^{\frac{1}{2}})$), where

$$\langle . \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega . \mathrm{d}x,$$

being understood that, for $v \in H^{-1}(\Omega)$,

$$\langle v \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle v, 1 \rangle_{H^{-1}(\Omega), H^1_{per}(\Omega)},$$

and we note that

$$v \mapsto (\|v - \langle v\rangle\|^2_{-1} + \langle v\rangle^2)^{\frac{1}{2}}$$

is a norm on $H^{-1}(\Omega)$ which is equivalent to the usual one. Here, $\Omega = \prod_{i=1}^{n}(0, L_i)$, $L_i > 0$, $n = 2$ or $n = 3$. Furthermore, for a space W we shall dénote by $\dot{W}$ the space

$$\dot{W} = \{v \in W,\ \langle v \rangle = 0\}.$$

Remark 2.1. *Actually, the conseved phase-field system usually is endowed with Neumann boundary conditions. In our case, these conditions read*

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} \quad (= \frac{\partial A u}{\partial \nu}) = \frac{\partial \alpha}{\partial \nu} = 0 \quad on \quad \Gamma, \tag{2.7}$$

where ν denotes the unit outer normal.

Ref

[18] J.E. Taylor, Mean curvature and weighted mean curvature, Acta Metall. Mater., 1992, 40, 1475–1495.

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Remark 2.2. *Note that similar properties hold for the operator $-\Delta$, with obvious changes.*

Having this, we rewrite (2.1) as

$$\frac{\partial u}{\partial t} - \Delta A u - \Delta f(u) = -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right). \tag{2.8}$$

Furthermore, we assume that the function f satisfies the following conditions:

$$f \in C^2(\mathbb{R}), \quad f(0) = 0, \tag{2.9}$$

$$f' \geqslant -c_0, \quad c_0 \geqslant 0, \tag{2.10}$$

$$f(s)s \geqslant c_1 F(s) - c_2, \quad F(s) \geqslant -c_3, \quad c_1 > 0, \quad c_2, c_3 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.11}$$

where, we denote by F the primitive of f vanishing at $s = 0$,

$$c_4 s^{2p-1} - c_5 \leqslant f''(s) \leqslant c_6 s^{2p-1} + c_7, \quad c_4, c_6 > 0, \quad c_5, c_7 \geqslant 0, \quad p \geqslant 1, \quad s \in \mathbb{R}. \tag{2.12}$$

Remark 2.3. *In particular, these assumptions are satisfied by function*

$$f(s) = \sum_{i=1}^{2p+1} a_i s^i, \quad a_{2p+1} > 0, \quad \forall s \in \mathbb{R}$$

(and, the usual cubic nonlinear term $f(s) = s^3 - s$).

Throughout the paper, the same letters c, c' and c'' denote (generally positive) constants which may vary from line to line. Similaly, the same letter Q denotes (positive) monotone increasing (with respect to each argument) and continuous functions which may vary from line to line.

III. A Priori Estimates

The estimates below are formal, but they can also be justified within a Galerkin scheme for the approximated problem.

We first note that, integrating (formally) (2.8) over Ω, we have

$$\frac{d\langle u\rangle}{dt} = 0,$$

hence

$$\langle u(t)\rangle = \langle u_0\rangle, \quad \forall t \geqslant 0. \tag{3.1}$$

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Furthermore, integrating (2.2) over Ω, we obtain, in view of (2.7),

$$\frac{d^2\langle\alpha\rangle}{dt^2} = -\frac{d\langle u\rangle}{dt}. \tag{3.2}$$

It thus follows from (2.4) and (3.2) that

$$\frac{d\langle\alpha\rangle}{dt} = \langle u_0 + \alpha_1\rangle - \langle u\rangle, \tag{3.3}$$

meaning, in particular, that $\left\langle u + \frac{\partial\alpha}{\partial t}\right\rangle$ is a conserved quantity and from (3.1) that

$$\frac{d\langle\alpha\rangle}{dt} = \langle\alpha_1\rangle, \tag{3.4}$$

so that

$$\langle\alpha(t)\rangle = \langle\alpha_0\rangle + \langle\alpha_1\rangle t, \quad t \geqslant 0. \tag{3.5}$$

We now assume that

$$|\langle u_0\rangle| \leqslant M_1, \quad |\langle\alpha_1\rangle| \leqslant M_2, \quad |\langle u_0 + \alpha_1\rangle| \leqslant M_1 + M_2, \tag{3.6}$$

for fixed positive constants M_1 et M_2. Thus,

$$|\langle u(t)\rangle| \leqslant M_1, \quad \left|\left\langle\frac{\partial\alpha}{\partial t}(t)\right\rangle\right| \leqslant M_2, \quad \left|\left\langle u + \frac{\partial\alpha}{\partial t}\right\rangle(t)\right| \leqslant M_1 + M_2, \quad t \geqslant 0. \tag{3.7}$$

Furthemore, it follows from (3.5) that

$$|\langle\alpha(t)\rangle| \leqslant |\langle\alpha_0\rangle| + |\langle\alpha_1\rangle| t, \quad t \geqslant 0. \tag{3.8}$$

We rewrite, in view of (3.4), (2.8) as

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} + Au + f(u) - \langle f(u)\rangle = \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t} + \langle\alpha_1\rangle \tag{3.9}$$

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and, in view of (3.3) an (3.9) that

$$(-\Delta)^{-1}\frac{\partial u}{\partial t}+Au+f(u)-\langle f(u)\rangle=\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}+\langle\alpha_1\rangle. \tag{3.10}$$

Notes

Furthermore, we deduce from (2.2) and (3.2) that

$$\frac{\partial^2\overline{\alpha}}{\partial t^2}-\Delta\frac{\partial^2\overline{\alpha}}{\partial t^2}-\Delta\frac{\partial\overline{\alpha}}{\partial t}-\Delta\overline{\alpha}=-\frac{\partial\overline{u}}{\partial t}, \tag{3.11}$$

We first multiply (3.9) by $\frac{\partial u}{\partial t}$ and obtain, noting that $\left\langle\frac{\partial u}{\partial t}\right\rangle=0$,

$$\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{2}}u\|^2+2\int_\Omega F(u)\mathrm{d}x)+\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2=\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},\frac{\partial u}{\partial t}\right)\right). \tag{3.12}$$

We then multiply (2.2) by $\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\alpha\|^2+\|\Delta\alpha\|^2+\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right)+\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2$$

$$=-\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},\frac{\partial u}{\partial t}\right)\right) \tag{3.13}$$

(note indeed that $\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|^2=\left\|\frac{\partial\alpha}{\partial t}\right\|^2+2\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2$).

Summing finally (3.12) and (3.13), we find a differential equality

$$\frac{dE_1}{dt}+2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+2\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2+2\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2=0 \tag{3.14}$$

where

$$E_1=\|A^{\frac{1}{2}}u\|^2+2\int_\Omega F(u)\mathrm{d}x+\|\nabla\alpha\|^2+\|\Delta\alpha\|^2+\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|^2$$

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satisfies, owing to (2.12),

$$c\left(\|A^{\frac{1}{2}}u\|^2+\|u\|^{2p+2}_{L^{2p+2}(\Omega)}+\|\Delta\alpha\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right)+c'\leqslant E_1$$

$$\leqslant c''\left(\|A^{\frac{1}{2}}u\|^2+\|u\|^{2p+2}_{L^{2p+2}(\Omega)}+\|\Delta\alpha\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right)+c''',\quad c,c''>0. \tag{3.15}$$

(here and below, when not specified, the sign of the constants (c' and c''' here) can be arbitrary).

Multiplying (3.9) by $\overline{u}$ and have, integrating over Ω and by parts,

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}\|^2_{-1}+\|A^{\frac{1}{2}}u\|^2+((f(u),u))=\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},u\right)\right)+((f(u),\langle u\rangle))-\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},\langle u\rangle\right)\right).$$

It follows from (2.11) that

$$((f(u),u))\geqslant c_2\int_\Omega F(u)\mathrm{d}x+c,$$

from (2.12) and (3.7) that

$$|((f(u),\langle u\rangle))|\leqslant cM_1\int_\Omega|f(u)|\mathrm{d}x\leqslant\frac{c_2}{2}\int_\Omega F(u)\mathrm{d}x+c_{M_1}$$

and from (3.7) that

$$\left|\left(\left(\frac{\partial\alpha}{\partial t},\langle u\rangle\right)\right)\right|\leqslant c_{M_1,M_2}. \tag{3.16}$$

Therefore, owing again to (3.7) and remembering that $v\mapsto(\|A^{\frac{1}{2}}v\|^2+\langle v\rangle^2)^{\frac{1}{2}}$ is a norm in $H^1_{per}(\Omega)$ which is equivalent to the usual H^1-norm,

$$\frac{d}{dt}\|\overline{u}\|^2_{-1}+c(\|u\|^2_{H^1_{per}(\Omega)}+2\int_\Omega F(u)\mathrm{d}x)\leqslant c'\left(\|\frac{\partial\alpha}{\partial t}\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2\right)+c''_{M_1,M_2},\quad c>0. \tag{3.17}$$

Summing (3.14) and δ_1(3.17), where $\delta_1>0$ is small enough, we have a differential inequality of the form

$$\frac{dE_2}{dt}+c\left(\|u\|^2_{H^1_{per}(\Omega)}+2\int_\Omega F(u)\mathrm{d}x+\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}+\left\|\frac{\partial\alpha}{\partial t}\right\|^2_{H^2_{per}(\Omega)}\right)+\leqslant c'_{M_1,M_2},\quad c>0, \tag{3.18}$$

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where

$$E_2 = E_1 + \delta_1 \|\overline{u}\|_{-1}^2$$

satisfies

$$c\left(\|A^{\frac{1}{2}}u\|^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta\alpha\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + c' \leqslant E_2$$

$$\leqslant c''\left(\|A^{\frac{1}{2}}u\|^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta\alpha\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + c''', \quad c, c'' > 0. \tag{3.19}$$

We multiply (2.8) by u to obtain, owing to (2.9) and (2.10),

$$\frac{d}{dt}\|u\|^2 + \|\nabla A^{\frac{1}{2}}u\|^2 \leqslant c\left(\|u\|_{H_{per}^1(\Omega)}^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right). \tag{3.20}$$

Summing (3.18) and δ_2(3.20), where $\delta_2 > 0$ is small enough, we obtain a differential inequality of the form

$$\frac{dE_3}{dt} + c\left(\|u\|_{H_{per}^2(\Omega)}^2 + 2\int_\Omega F(u)\mathrm{d}x + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|_{H_{per}^2(\Omega)}^2\right) \leqslant c'_{M_1,M_2}, \quad c > 0, \tag{3.21}$$

where

$$E_3 = E_2 + \delta_2\|u\|^2$$

satisfies

$$c\left(\|u\|_{H_{per}^1(\Omega)}^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta\alpha\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + c' \leqslant E_3$$

$$\leqslant c''\left(\|u\|_{H_{per}^1(\Omega)}^2 + \|u\|_{L^{2p+2}(\Omega)}^{2p+2} + \|\Delta\alpha\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right) + c''', \quad c, c'' > 0. \tag{3.22}$$

Now, multiplying (3.10) by $\frac{\partial u}{\partial t}$, we have

$$\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{2}}u\|^2 + 2\int_\Omega F(u)\mathrm{d}x) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 = \left(\left(\frac{\partial\overline{\alpha}}{\partial t} - \Delta\frac{\partial\overline{\alpha}}{\partial t}, \frac{\partial\overline{u}}{\partial t}\right)\right). \tag{3.23}$$

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We then multiply (3.11) by $\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}$

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\overline{\alpha}\|^2+\|\Delta\overline{\alpha}\|^2+\left\|\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2\right)+\left\|\nabla\frac{\partial\overline{\alpha}}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2$$

$$=-\left(\left(\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t},\frac{\partial\overline{u}}{\partial t}\right)\right) \tag{3.24}$$

(note indeed that $\left\|\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2=\left\|\frac{\partial\overline{\alpha}}{\partial t}\right\|^2+2\left\|\nabla\frac{\partial\overline{\alpha}}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2$).

Summing finally (3.21), (3.23) and (3.24), we obtaine a differential inequality of the form

$$\frac{dE_4}{dt}+c\left(\|u\|^2_{H^2_{per}(\Omega)}+2\int_\Omega F(u)\mathrm{d}x+\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}+\left\|\frac{\partial\alpha}{\partial t}\right\|^2_{H^2_{per}(\Omega)}+\left\|\frac{\partial\overline{\alpha}}{\partial t}\right\|^2_{H^2_{per}(\Omega)}\right)\leqslant c'_{M_1,M_2},\quad c>0, \tag{3.25}$$

where

$$E_4=E_3+\|A^{\frac{1}{2}}u\|^2+2\int_\Omega F(u)\mathrm{d}x+\|\nabla\overline{\alpha}\|^2+\|\Delta\overline{\alpha}\|^2+\left\|\frac{\partial\overline{\alpha}}{\partial t}-\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2$$

satisfies

$$c\left(\|u\|^2_{H^1_{per}(\Omega)}+\|u\|^{2p+2}_{L^{2p+2}(\Omega)}+\|\Delta\alpha\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2+\|\Delta\overline{\alpha}\|^2+\left\|\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2\right)+c'\leqslant E_4$$

$$\leqslant c''\left(\|u\|^2_{H^1_{per}(\Omega)}+\|u\|^{2p+2}_{L^{2p+2}(\Omega)}+\|\Delta\alpha\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2+\|\Delta\overline{\alpha}\|^2+\left\|\Delta\frac{\partial\overline{\alpha}}{\partial t}\right\|^2\right)+c''',\quad c,c''>0. \tag{3.26}$$

We finally assume that $p=1$ when $n=3$ and multiply (2.8) by Au to find

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2+\|\nabla Au\|^2+((f'(u)\nabla u,\nabla Au))=\left(\left(\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t},\nabla Au\right)\right).$$

We assume that $n=3$ and $p=1$ (the case $n=2$ can be treated in a similar way) and have, owing to (2.12) and Hölder's inequality,

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$$|((f'(u)\nabla u, \nabla Au))| \leqslant c\int_\Omega (|u|^2+1)|\nabla u||\nabla Au|\mathrm{d}x$$

$$\leqslant c(\|u\|^2_{L^6(\Omega)}+1)\|\nabla u\|_{L^6(\Omega)}\|\nabla Au\| \leqslant c(\|u\|^2_{H^1_{per}(\Omega)}+1)\|u\|_{H^2_{per}(\Omega)}\|\nabla Au\|.$$

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Therefore,

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2+\|\nabla Au\|^2 \leqslant c(\|u\|^4_{H^1_{per}(\Omega)}+1)\|u\|^2_{H^2_{per}(\Omega)}+c'\left(\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2+\left\|\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right). \quad (3.27)$$

IV. Well-Posedness

We have the following result.

Theorem 4.1. We assume that (2.9)–(2.12) hold. Then, for every $(u_0,\alpha_0,\alpha_1) \in (H^1_{per}(\Omega)\cap L^{2p+2}(\Omega))\times H^2_{per}(\Omega)\times H^2_{per}(\Omega))$, *(2.1)–(2.4) possesses at least one solution* $(u,\alpha,\frac{\partial\alpha}{\partial t})$ *such that*

$$u \in L^\infty(0,T;H^1_{per}(\Omega)\cap L^{2p+2}(\Omega))\cap L^2(0,T;H^2_{per}(\Omega)),$$

$$\frac{\partial u}{\partial t}\in L^2(0,T;H^{-1}(\Omega)),$$

$$\alpha,\overline{\alpha}\in L^\infty(0,T;H^2_{per}(\Omega))$$

and

$$\frac{\partial\alpha}{\partial t},\frac{\partial\overline{\alpha}}{\partial t}\in L^\infty(0,T;H^2_{per}(\Omega))\cap L^2(0,T;H^2_{per}(\Omega))$$

$\forall T>0.$

Furthermore, if $p=1$ *when* $n=3$, *then*

$$u\in L^2(0,T;H^3_{per}(\Omega)).$$

Proof. The proof is based on (3.8), (3.25), (3.27) and, e.g., a standard Galerkin scheme. □

We have, concerning the uniqueness, the following.

Theorem 4.2. We assume that the assumptions of Theorem 4.1 hold and that $p=1$ *when* $n=3$ *and* $p\in[1,2]$ *when* $n=2$. *Then, the solution obtained in Theorem 4.1 is unique.*

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Proof. Let $\left(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right)$ and $\left(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$ be two solutions to (2.1)–(2.3) with initial data $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$, respectively, such that

$$|\langle u_0^{(i)} \rangle| \leqslant M_1, \quad |\langle \alpha_1^{(i)} \rangle| \leqslant M_2, \quad |\langle u_0^{(i)} + \alpha_1^{(i)} \rangle| \leqslant M_1 + M_2, \quad i = 1, 2, \tag{4.1}$$

for fixed positive constants M_1 and M_2. We set

$$\left(u, \alpha, \frac{\partial \alpha}{\partial t}\right) = \left(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right) - \left(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) - (u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}).$$

We have

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} + Au + f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)}) \rangle = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} + \langle \alpha_1 \rangle, \tag{4.2}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{4.3}$$

$$u \quad \text{and} \quad \alpha \quad \text{are} \quad \Omega - \text{periodic}, \tag{4.4}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{4.5}$$

We multiply (4.2) by $\frac{\partial u}{\partial t}$ (note that $\left\langle \frac{\partial u}{\partial t} \right\rangle = 0$) and obtain

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u\|^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2 = \left(\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t} \right) \right) - \left(\left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right). \tag{4.6}$$

We first assume that $n = 3$ and $p = 1$. We have

$$\left| \left(\left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right) \right| = \left| \left(\left(f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)}) \rangle, \frac{\partial u}{\partial t} \right) \right) \right|$$

$$\leqslant \|\nabla(f(u^{(1)})-f(u^{(2)}))\| \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}$$

$$= \Big\|\nabla(\int_0^1 f'(u^{(1)}+s(u^{(2)}-u^{(1)}))\mathrm{d}su)\Big\| \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}$$

$$\leqslant \Big\|\int_0^1 f'(u^{(1)}+s(u^{(2)}-u^{(1)}))\mathrm{d}s\nabla u\Big\| \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}$$

$$+ \Big\|u\int_0^1 f''(u^{(1)}+s(u^{(2)}-u^{(1)}))(\nabla u^{(1)}+s\nabla(u^{(2)}-u^{(1)}))\mathrm{d}s\Big\| \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}.$$

Furthermore, owing to Agmon's inequality,

$$\Big\|\int_0^1 f'(u^{(1)}+s(u^{(2)}-u^{(1)}))\mathrm{d}s\nabla u\Big\|^2 \leqslant c\int_\Omega(|u^{(1)}|^4+|u^{(2)}|^4+1)|\nabla u|^2\mathrm{d}x$$

$$\leqslant c(\|u^{(1)}\|^4_{L^\infty(\Omega)}+\|u^{(2)}\|^4_{L^\infty(\Omega)}+1)\|\nabla u\|^2$$

$$\leqslant c(\|u^{(1)}\|^2_{H^1_{per}(\Omega)}+\|u^{(2)}\|^2_{H^1_{per}(\Omega)}+1)$$

$$\times(\|u^{(1)}\|^2_{H^2_{per}(\Omega)}+\|u^{(2)}\|^2_{H^2_{per}(\Omega)}+1)\|\nabla u\|^2$$

and, owing to Hölder's inequality,

$$\Big\|u\int_0^1 f''(u^{(1)}+s(u^{(2)}-u^{(1)}))(\nabla u^{(1)}+s\nabla(u^{(2)}-u^{(1)}))\mathrm{d}s\Big\|^2$$

$$\leqslant c\int_\Omega(|u^{(1)}|^2+|u^{(2)}|^2+1)(|\nabla u^{(1)}|^2+|\nabla u^{(2)}|^2)|u|^2\mathrm{d}x$$

$$\leqslant c(\|u^{(1)}\|^2_{H^1_{per}(\Omega)}+\|u^{(2)}\|^2_{H^1_{per}(\Omega)}+1)(\|u^{(1)}\|^2_{H^2_{per}(\Omega)}+\|u^{(2)}\|^2_{H^2_{per}(\Omega)})\|u\|^2_{H^1_{per}(\Omega)}.$$

We now assume that $n=2$ and we take the most complicated case $p=2$. Then, owing to Agmon's inequality and a proper interpolation inequality,

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$$\Big\|\int_0^1 f'(u^{(1)}+s(u^{(2)}-u^{(1)}))\mathrm{d}s\nabla u\Big\|^2 \leqslant c\int_\Omega(|u^{(1)}|^8+|u^{(2)}|^8+1)|\nabla u|^2\mathrm{d}x$$

$$\leqslant c(\|u^{(1)}\|^8_{L^\infty(\Omega)}+\|u^{(2)}\|^8_{L^\infty(\Omega)}+1)\|\nabla u\|^2$$

$$\leqslant c(\|u^{(1)}\|^4\|u^{(1)}\|^4_{H^2_{per}(\Omega)}+\|u^{(1)}\|^4\|u^{(1)}\|^4_{H^2_{per}(\Omega)}+1)\|\nabla u\|^2$$

$$\leqslant c(\|u^{(1)}\|^6_{H^1_{per}(\Omega)}+\|u^{(2)}\|^6_{H^1_{per}(\Omega)}+1)(\|u^{(1)}\|^2_{H^3_{per}(\Omega)}$$

$$+\|u^{(2)}\|^2_{H^3_{per}(\Omega)}+1)\|\nabla u\|^2.$$

Furthermore, owing to Hölder's inequality,

$$\Big\|u\int_0^1 f''(u^{(1)}+s(u^{(2)}-u^{(1)}))(\nabla u^{(1)}+s\nabla(u^{(2)}-u^{(1)}))\mathrm{d}s\Big\|^2$$

$$\leqslant c\int_\Omega(|u^{(1)}|^6+|u^{(2)}|^6+1)(|\nabla u^{(1)}|^2+|\nabla u^{(2)}|^2)|u|^2\mathrm{d}x$$

$$\leqslant c(\|u^{(1)}\|^6_{H^1_{per}(\Omega)}+\|u^{(2)}\|^6_{H^1_{per}(\Omega)}+1)(\|u^{(1)}\|^2_{H^2_{per}(\Omega)}+\|u^{(2)}\|^2_{H^1_{per}(\Omega)})\|u\|^2_{H^1_{per}(\Omega)}.$$

Finally, we obtain, in both cases, an inequality of the form

$$\frac{d}{dt}\|A^{\frac12}u\|^2+\|\frac{\partial u}{\partial t}\|^2_{-1}\leqslant 2\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},\frac{\partial u}{\partial t}\right)\right)$$

$$+c(\|u^{(1)}\|^q_{H^1_{per}(\Omega)}+\|u^{(2)}\|^q_{H^1_{per}(\Omega)}+1)(\|u^{(1)}\|^2_{H^3_{per}(\Omega)}+\|u^{(2)}\|^2_{H^3_{per}(\Omega)}+1)\|\nabla u\|^2,\quad q\geqslant 1. \quad (4.7)$$

Multiplying then (4.3) by $\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}$, we find

$$\frac{d}{dt}\left(\|\nabla\alpha\|^2+\|\Delta\alpha\|^2+\Big\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\Big\|^2\right)+2\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2+2\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2$$

$$=-2\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},\frac{\partial u}{\partial t}\right)\right). \quad (4.8)$$

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Summing finally (4.7) and (4.8), we have, an inequality of the form

$$\frac{dE_5}{dt} \leqslant c(\|u^{(1)}\|^q_{H^1_{per}(\Omega)} + \|u^{(2)}\|^q_{H^1_{per}(\Omega)} + 1)(\|u^{(1)}\|^2_{H^3_{per}(\Omega)} + \|u^{(2)}\|^2_{H^3_{per}(\Omega)} + 1)E_4, \qquad (4.9)$$

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$q \geqslant 1$, where

$$E_5 = \|A^{\frac{1}{2}}u\|^2 + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|^2.$$

Then, We deduce from (4.9), (3.8), the estimates obtained in the previous subsection and Gronwall's lemma the uniqueness, as well as the continuous dependence with respect to the initial data. □

V. Regularity of Solutions

We have the following result which gives the existence and uniqueness of more regular solutions.

Theorem 5.1. We assume that the assumptions of Theorem 4.1 hold and that (2.12) is replaced by

$$|f(s)| \leqslant \epsilon F(s) + c_\epsilon, \quad \forall\epsilon > 0, \quad s \in \mathbb{R}. \qquad (5.1)$$

Then, if $(u_0, \alpha_0, \alpha_1) \in H^2_{per}(\Omega) \times H^3_{per}(\Omega) \times H^3_{per}(\Omega)$, *the problem possesses a unique solution such that*

$$u \in L^\infty(0, T; H^2_{per}(\Omega))$$

and

$$\alpha, \frac{\partial\alpha}{\partial t} \in L^\infty(0, T; H^3_{per}(\Omega)), \quad \forall T > 0.$$

Proof. The proof of uniqueness is obtained by proceeding as in that of Theorem 4.2, noting that, with the higher regularity considered here, no growth assumption on f is needed, owing to the continuous embedding $H^2_{per}(\Omega) \subset L^\infty(\Omega)$.

We now turn to the proof of existence and, more precisely, of the further regularity of the solutions.

We multiply (2.8) by $\frac{\partial u}{\partial t}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla A^{\frac{1}{2}}u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 = \left(\left(\Delta f(u), \frac{\partial u}{\partial t}\right)\right) + \left(\left(\nabla\frac{\partial u}{\partial t}, \nabla\frac{\partial\alpha}{\partial t} - \nabla\Delta\frac{\partial\alpha}{\partial t}\right)\right). \qquad (5.2)$$

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Multiplying then (2.2) by $-\Delta\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\left\|\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right)+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2+\left\|\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2$$

$$=-\left(\left(\nabla\frac{\partial u}{\partial t},\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t}\right)\right). \tag{5.3}$$

Summing finally (5.2) and (5.3), we find

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla A^{\frac{1}{2}}u\|^2+\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\left\|\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right)$$

$$+\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2+\left\|\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2=\left(\left(\Delta f(u),\frac{\partial u}{\partial t}\right)\right). \tag{5.4}$$

It follows from (5.4) and the continuous embedding $H^2_{per}(\Omega)\subset\mathcal{C}(\overline{\Omega})$ that

$$\frac{d}{dt}\left(\|\nabla A^{\frac{1}{2}}u\|^2+\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\left\|\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2\right)$$

$$+\left\|\frac{\partial u}{\partial t}\right\|^2+2\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2+2\left\|\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2\leqslant Q(\|\Delta u\|^2). \tag{5.5}$$

We set

$$y=\|\nabla A^{\frac{1}{2}}u\|^2+\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\left\|\nabla\frac{\partial\alpha}{\partial t}-\nabla\Delta\frac{\partial\alpha}{\partial t}\right\|^2 \tag{5.6}$$

and we deduce from (5.5) that we have an inequality of the form

$$y'\leqslant Q(y). \tag{5.7}$$

Let z be the solution to the ordinary differential equation

$$z'=Q(z),\quad z(0)=y(0). \tag{5.8}$$

It follows from the comparison principle that there exists $T_0=T_0(\|u_0\|_{H^2_{per}(\Omega)},\|\alpha_0\|_{H^3_{per}(\Omega)},\|\alpha_1\|_{H^3_{per}(\Omega)})$ (say, belongin to $(0,\frac{1}{2})$) such that

$$y(t)\leqslant z(t),\quad t\in[0,T_0], \tag{5.9}$$

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from which it follows, owing also to (3.8) and (3.25), that

$$\|u(t)\|^2_{H^2_{per}(\Omega)} + \|\alpha(t)\|^2_{H^3_{per}(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^3_{per}(\Omega)}$$

$$\leqslant Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \in [0,T_0]. \tag{5.10}$$

We now differentiate (3.9) with respect to times and have, owing to (2.2),

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} - \left\langle f'(u)\frac{\partial u}{\partial t}\right\rangle = \Delta\alpha + \Delta\frac{\partial \alpha}{\partial t} - \frac{\partial u}{\partial t}. \tag{5.11}$$

We multiply (5.11) by $t\frac{\partial u}{\partial t}$ and have

$$\frac{1}{2}\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right) + t\left\|A^{\frac{1}{2}}\frac{\partial u}{\partial t}\right\|^2 + t\left(\left(f'(u)\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right)\right)$$

$$= -t\left(\left(\nabla\alpha, \nabla\frac{\partial u}{\partial t}\right)\right) - t\left(\left(\nabla\frac{\partial \alpha}{\partial t}, \nabla\frac{\partial u}{\partial t}\right)\right) - t\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2_{-1},$$

which yields, owing to (2.10) and a proper interpolation inequality (see the proof of Theorem 4.2),

$$\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right) + ct\left\|\frac{\partial u}{\partial t}\right\|^2_{H^1_{per}(\Omega)} \leqslant c't\left(\|\nabla\alpha\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right) + \left\|\frac{\partial u}{\partial t}\right\|^2_{-1}, \quad c > 0. \tag{5.12}$$

It follows from (3.25), (5.12) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|^2_{-1} \leqslant \frac{1}{t}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \in (0,T_0]. \tag{5.13}$$

Next, we multiply (5.11) by $\frac{\partial u}{\partial t}$ and obtain, proceeding similarly,

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + c\left\|\frac{\partial u}{\partial t}\right\|^2_{H^1_{per}(\Omega)} \leqslant c'\left(\|\nabla\alpha\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right), \quad c > 0. \tag{5.14}$$

We deduce from (3.25), (5.14) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|^2_{-1} \leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)})\left\|\frac{\partial u}{\partial t}(T_0)\right\|^2_{-1}, \quad t \geqslant T_0,$$

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hence, owing to (5.13),

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leqslant e^{ct} Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant T_0. \tag{5.15}$$

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We rewrite, for $t \geqslant T_0$ fixed, (3.9) as an elliptic equation,

$$Au + f(u) - \langle f(u) \rangle = h_u(t), \quad u \quad \text{is} \quad \Omega - \text{periodic}, \tag{5.16}$$

where

$$h_u(t) = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} + \langle \alpha_1 \rangle \tag{5.17}$$

satisfies, owing to (3.25) and (5.15),

$$\|h_u(t)\| \leqslant e^{ct} Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant T_0. \tag{5.18}$$

Multiplying (5.16) by $\overline{u}$, we find, owing to (2.11) and (5.1),

$$\|A^{\frac{1}{2}}u(t)\|^2 + c\int_\Omega F(u(t))\mathrm{d}x \leqslant c'_{M_1}(\|h_u(t)\|^2 + 1), \quad c > 0, \quad t \geqslant T_0. \tag{5.19}$$

Multiplying then (5.16) by Au, we have, owing to (2.10),

$$\|Au(t)\|^2 \leqslant c(\|A^{\frac{1}{2}}u(t)\|^2 + \|h_u(t)\|^2), \quad t \geqslant T_0. \tag{5.20}$$

Combining (5.19) and (5.20), we finally obtain, owing to (3.25) and (5.18),

$$\|u(t)\|^2_{H^2_{per}(\Omega)} \leqslant e^{ct} Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant T_0, \tag{5.21}$$

hence, owing to (5.10),

$$\|u(t)\|^2_{H^2_{per}(\Omega)} \leqslant e^{ct} Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant 0. \tag{5.22}$$

We now come back to (5.3), from which it follows that

$$\frac{d}{dt}\left(\|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t} - \nabla\Delta\frac{\partial \alpha}{\partial t}\right\|^2\right) \leqslant \left\|\nabla\frac{\partial u}{\partial t}\right\|^2. \tag{5.23}$$

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Noting that it follows from (3.25), (5.14) and (5.15) that

$$\int_{T_0}^{t}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \mathrm{d}\tau \leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant T_0, \tag{5.24}$$

we deduce from (3.14), (5.23) and (5.24) that

$$\|\Delta\alpha(t)\|^2 + \|\nabla\Delta\alpha(t)\|^2 + \left\|\left(\nabla\frac{\partial\alpha}{\partial t} - \nabla\Delta\frac{\partial\alpha}{\partial t}\right)(t)\right\|^2$$

$$\leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)})$$

$$+\|\Delta\alpha(T_0)\|^2 + \|\nabla\Delta\alpha(T_0)\|^2 + \left\|\left(\nabla\frac{\partial\alpha}{\partial t} - \nabla\Delta\frac{\partial\alpha}{\partial t}\right)(T_0)\right\|^2, \quad t \geqslant T_0,$$

hence, owing to (3.8), (3.25), (5.10) and (5.22),

$$\|u(t)\|^2_{H^2_{per}(\Omega)} + \|\alpha(t)\|^2_{H^3_{per}(\Omega)} + \left\|\frac{\partial\alpha}{\partial t}(t)\right\|^2_{H^3_{per}(\Omega)}$$

$$\leqslant e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2_{per}(\Omega)}, \|\alpha_0\|_{H^3_{per}(\Omega)}, \|\alpha_1\|_{H^3_{per}(\Omega)}), \quad t \geqslant 0, \tag{5.25}$$

which finishes the proof of the theorem. □

Remark 5.1. We can note that, here, we are not able to study the asymptotic behavior of the associated dynamical system. Indeed, the estimates derived in this section are not dissipative.

Acknowledgements

The author is thankful to Alain Miranville for helpful discussions. He also wishes to thank an anonymous referee for her/his careful reading of the paper and useful comments.

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Notes

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