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Exploring Torus Black-Holes In (1+3)-Dimensions: A Novel Approach to Higher Genus Solution

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Exploring Torus Black-Holes In (1+3)-Dimensions: A Novel Approach to Higher Genus Solution

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Abstract- A torus black-hole solution of the vacuum gravitational field equation of general relativity in (1 + 3)-dimensions is obtained. Starting with a metric ansatz associated with the torus, our method is based on straightforward computations the usual geometric mathematical tools of the Christoffel symbols and the Riemann tensor. Specifically, after deriving such mathematical tools the field equations of general relativity are considered. The resultatning equations are properly combained to find the solution. Moreover, the novelty and potential implications of this solution emerges from the fact that is based on a coordinate transformation metric ansatz. This provides with broad implications and future research directions. In particular we argue that our formalism can properly be used for a search of higher genus black-hole solution.

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I. Introduction

Traditionally black-holes are associated with a 2-dimensional sphere, S^2, called its event horizon, which defines the boundary where not even light can escape. However, in 2-dimensional space the sphere S^2 is just a particular case of compact simple connect manifolds. These manifolds are classified according to their genus g [1]. A S^2 corresponds to just $g = 0$ and for a donut or torus we have $g = 1$, and so on. Thus, from this mathematical perspective there is not any particular reason why to choose $g = 0$ for a black-hole system, rather than $g = 1$ or any other 2-dimensional compact simple connected manifold of arbitrary g. Physically, there are a large kind of torus-like black-holes [2]. In particular, several studies of thermodynamic torus-like black-hole have realized, including fluctuations, statistical entropy [3], the quantum effect on Hawking radiation [4], thermal fluctuations and quasi-normal modes [5], thermodynamic instability [6], Gibbs free energy [7], variation of the chaos bound in two regions [8], and weak cosmic censorship conjecture [9]. Also there have been much interest in topological aspects on torus-like black hole: dimensional black holes with toroidal or higher genus horizons [10], Born-Infeld-dilaton black holes [11] and topological black holes in anti-de Sitter space [12] (it may

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Ref

1. C. Nash and S. Sen, Topology and Geometry for Physicists (Academic Press INC, 1983).

be helpful to see also Ref. [13]-[15] and references therein). However, all of these developments have as a basic inspiration the 2-dimensional sphere. Of course, there are already examples of a 3-dimensional black-hole associated with S^3 event horizon (see Ref. [16] and references therein). But again the situation is very similar to the case of 2-sphere or S^2.

Here, for the above reasons we ask ourselves weather a torus black hole is possible, with a straightforward derivation that may be useful for another values of g, other than $g = 0$ and $g = 1$. In the case of $g = 1$ we have the $S^1 \times S^1$ topology. So, in this work we shall try to solve the general relativity field equations by proposing an ansatz metric which provides an alternative derivation for both 2-sphere of black-hole and a torus black-hole. We think that our work may be useful for studying another higher dimensional topologies for black-holes.

Since our formalism explore the possibility of torus black holes beyond the traditional 2-dimensional sphere event horizon, there are a few areas of research that could be improved:

I. Although, the previous paragraphs provides a general idea of the work's objectives, it remains to explain the progression of ideas considered in our formalism. In fact, starting with a metric ansatz associated with the torus coordinates, our method is based on straightforward computations the usual geometric mathematical tools of the Christoffel symbols and the Riemann tensor. These mathematical computations are substituted in the field equations of general relativity. The resultants equations are properly combined to find the solution for $g = 1$. This procedure opens the possibility to apply our method to higher genus.

II. Our work may help to have better understanding of the thermodynamic instability and weak cosmic censorship conjecture on black-hole physics. This is because our formalism may open new routes to investigate alternative topologies.

Technically, we organize this work as follows: In section 2, we propose the ansatz which must be substitute in the gravitational field equations. For this purpose, for such ansatz, we compute the Christoffel symbols and the Riemann tensor. The corresponding results are substitute in the vacuum gravitational field equations. In section 3, using the resulting field equations we start to propose the solution of a torus black-hole solution. Our result is analyzed and proved that in a specific limit is reduced to the traditional black-hole solution. Finally, in section 4, we comment how our procedure for genus $g = 0$ and $g = 1$ can be generalized to arbitrary genus g.

II. Ansatz

Consider the line element

$$ds^2 = -e^{f(r,\theta)}dt^2 + e^{h(r,\theta)}dr^2 + e^{q(r,\theta)}d\theta^2 + e^{p(r,\theta)}d\phi^2, \tag{1}$$

Ref

16. A. Alaee and H. K. Kunduri, J. Geom. Anal. 33 (2023) 231: 2205.09737 [gr-qc].

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which is appropriate for torus black-hole solution. The metric tensor, or ansatz, associated with (1) is given by the matrix

$$g_{\mu\nu} = \begin{pmatrix} -e^{f(r,\theta)} & 0 & 0 & 0 \\ 0 & e^{h(r,\theta)} & 0 & 0 \\ 0 & 0 & e^{q(r,\theta)} & 0 \\ 0 & 0 & 0 & e^{p(r,\theta)} \end{pmatrix}, \quad (2)$$

with inverse

$$g^{\mu\nu} = \begin{pmatrix} -e^{-f(r,\theta)} & 0 & 0 & 0 \\ 0 & e^{-h(r,\theta)} & 0 & 0 \\ 0 & 0 & e^{-q(r,\theta)} & 0 \\ 0 & 0 & 0 & e^{-p(r,\theta)} \end{pmatrix}. \quad (3)$$

Thus, the non-vanishing Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right\} = \Gamma^{\mu}_{\beta\alpha} \quad (4)$$

associated with (2) are

$$\Gamma^1_{12} = \tfrac{f'}{2}, \qquad \Gamma^2_{22} = \tfrac{h'}{2}, \qquad \Gamma^2_{11} = \tfrac{e^{f-h} f'}{2},$$
$$\Gamma^2_{33} = -\tfrac{e^{q-h} q'}{2}, \quad \Gamma^2_{44} = -\tfrac{e^{p-h} p'}{2}, \quad \Gamma^3_{32} = \tfrac{q'}{2}, \quad (5)$$
$$\Gamma^4_{42} = \tfrac{p'}{2},$$

and also

$$\Gamma^1_{13} = \tfrac{\dot{f}}{2}, \qquad \Gamma^2_{23} = \tfrac{\dot{h}}{2}, \qquad \Gamma^3_{11} = \tfrac{e^{f-q} \dot{f}}{2},$$
$$\Gamma^3_{22} = -\tfrac{e^{h-q} \dot{h}}{2}, \quad \Gamma^3_{44} = -\tfrac{e^{p-h} \dot{p}}{2}, \quad \Gamma^3_{33} = \tfrac{\dot{q}}{2}, \quad (6)$$
$$\Gamma^4_{43} = \tfrac{\dot{p}}{2}.$$

Here, we used the notations $F' \equiv \frac{\partial F}{\partial r}$ and $\dot{H} \equiv \frac{\partial F}{\partial \theta}$, for arbitrary functions $F = F(r,\theta)$ and $H = H(r,\theta)$. From these Christoffel symbols we may obtain the non-vanishing Riemann tensor

$$R^{\mu}_{\nu\alpha\beta} = \frac{\partial \Gamma^{\mu}_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\mu}_{\nu\alpha}}{\partial x^{\beta}} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha}. \tag{7}$$

In fact, we get the basic components:

$$R^{1}_{212} = -\frac{1}{2}f'' - \frac{1}{4}f'^2 + \frac{1}{4}f'h' - \frac{1}{4}\dot{f}\dot{h}e^{h-q}, \tag{8}$$

$$R^{1}_{313} = -\frac{1}{2}\ddot{f} - \frac{1}{4}\dot{f}^2 + \frac{1}{4}\dot{q}\dot{f} - \frac{1}{4}f'q'e^{q-h}, \tag{9}$$

$$R^{1}_{414} = -\frac{1}{4}f'p'e^{p-h} - \frac{1}{4}\dot{f}\dot{p}e^{p-q}, \tag{10}$$

$$R^{2}_{323} = -\frac{1}{2}\ddot{h} - \frac{1}{4}\dot{h}^2 + \frac{1}{4}\dot{h}\dot{q} - \frac{1}{2}q''e^{q-h} - \frac{1}{4}q'^2e^{q-h} + \frac{1}{4}h'q'e^{q-h}, \tag{11}$$

$$R^{2}_{424} = -\frac{1}{2}p''e^{p-h} - \frac{1}{4}p'^2e^{p-h} + \frac{1}{4}h'p'e^{p-h} - \frac{1}{4}\dot{h}\dot{p}e^{p-q}, \tag{12}$$

$$R^{3}_{434} = -\frac{1}{2}\ddot{p}e^{p-q} - \frac{1}{4}\dot{p}^2e^{p-q} + \frac{1}{4}\dot{p}\dot{q}e^{p-q} - \frac{1}{4}p'q'e^{p-h}. \tag{13}$$

In vacuum, the gravitational field equations can be written as [17]

$$R_{\mu\nu} = 0, \tag{14}$$

where $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ is the Ricci tensor. From (8), (9), (10) and (14), in a convenient arraignment, we get

$$R_{11} = \tfrac{1}{2}e^{f-h}(f'' + \tfrac{1}{2}f'^2 - \tfrac{1}{2}f'h' + \tfrac{1}{2}f'q' + \tfrac{1}{2}f'p')$$

$$+\tfrac{1}{2}e^{f-q}(\ddot{f} + \tfrac{1}{2}\dot{f}^2 - \tfrac{1}{2}\dot{f}\dot{q} + \tfrac{1}{2}\dot{f}\dot{h} + \tfrac{1}{2}\dot{f}\dot{p}) = 0, \tag{15}$$

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$$R_{22} = -\tfrac{1}{2}(f'' + \tfrac{1}{2}f'^2 - \tfrac{1}{2}f'h')$$

$$-\tfrac{1}{2}(p'' + \tfrac{1}{2}p'^2 - \tfrac{1}{2}p'h' + q'' + \tfrac{1}{2}q'^2 - \tfrac{1}{2}q'h') \tag{16}$$

$$-\tfrac{1}{2}e^{h-q}(\ddot{h} + \tfrac{1}{2}\dot{h}^2 - \tfrac{1}{2}\dot{h}\dot{q} + \tfrac{1}{2}\dot{h}\dot{f} + \tfrac{1}{2}\dot{h}\dot{p}) = 0,$$

$$R_{33} = -\tfrac{1}{2}e^{q-h}(q'' + \tfrac{1}{2}q'^2 - \tfrac{1}{2}q'h' + \tfrac{1}{2}q'f' + \tfrac{1}{2}q'p')$$

$$-\tfrac{1}{2}(\ddot{f} + \tfrac{1}{2}\dot{f}^2 - \tfrac{1}{2}\dot{f}\dot{q} + \ddot{h} + \tfrac{1}{2}\dot{h}^2 \tag{17}$$

$$-\tfrac{1}{2}\dot{h}\dot{q} + \ddot{p} + \tfrac{1}{2}\dot{p}^2 - \tfrac{1}{2}\dot{p}\dot{q}) = 0,$$

$$R_{44} = -\tfrac{1}{2}e^{p-h}(p'' + \tfrac{1}{2}p'^2 - \tfrac{1}{2}p'h' + \tfrac{1}{2}p'f' + \tfrac{1}{2}p'q')$$
$$-\tfrac{1}{2}e^{p-q}(\ddot{p} + \tfrac{1}{2}\dot{p}^2 - \tfrac{1}{2}\dot{p}\dot{q} + \tfrac{1}{2}\dot{p}\dot{h} + \tfrac{1}{2}\dot{p}\dot{f}) = 0. \tag{18}$$

Our main goal now is to solve (15)-(18) for the torus.

III. Torus Solution

For this purpose, first, it turns out reasonable to assume that

$$q'' + \frac{1}{2}q'^2 = 0 \tag{19}$$

and

$$p'' + \frac{1}{2}p'^2 = 0. \tag{20}$$

The reason for this it is because in both cases the general solution is of the form

$$e^{\frac{\xi}{2}} = rA_\xi(\theta) + B_\xi(\theta) \tag{21}$$

for $\xi = q$ or $\xi = p$. For the 2-sphere case we have $e^{\frac{q}{2}} = r$ and $e^{\frac{p}{2}} = r\sin\theta$. The choice $e^{\frac{q}{2}} = r$ implies that $A_q = 1$ and $B_q = 0$, while choosing $e^{\frac{p}{2}} = r\sin\theta$

means that $A_p = \sin\theta$ and $B_p = 0$. For the torus we have again $e^{\frac{q}{2}} = r$, but $e^{\frac{p}{2}} = r\sin\theta + a$ which means that $A_p = \sin\theta$ and $B_p = a$. Thus, considering (19) and (20) we get that (16), (17) and (18) simplify in the form

$$R_{22} = -\tfrac{1}{2}(f'' + \tfrac{1}{2}f'^2 - \tfrac{1}{2}f'h') + \tfrac{1}{4}h'(p' + q')$$
$$-\tfrac{1}{2}e^{h-q}(\ddot{h} + \tfrac{1}{2}\dot{h}^2 + \tfrac{1}{2}\dot{h}\dot{f} + \tfrac{1}{2}\dot{h}\dot{p}) = 0, \tag{22}$$

$$R_{33} = -\tfrac{1}{4}e^{q-h}(-q'h' + q'f' + q'p')$$
$$-\tfrac{1}{2}(\ddot{f} + \tfrac{1}{2}\dot{f}^2 + \ddot{h} + \tfrac{1}{2}\dot{h}^2 + \ddot{p} + \tfrac{1}{2}\dot{p}^2 - \tfrac{1}{2}\dot{p}\dot{q}) = 0, \tag{23}$$

and

$$R_{44} = -\tfrac{1}{4}e^{p-h}(-p'h' + p'f' + p'q')$$
$$-\tfrac{1}{2}e^{p-q}(\ddot{p} + \tfrac{1}{2}\dot{p}^2 + \tfrac{1}{2}\dot{p}\dot{h} + \tfrac{1}{2}\dot{p}\dot{f}) = 0, \tag{24}$$

where we also set $\dot{q} = 0$ because our choice $e^{\frac{q}{2}} = r$. The expression (15) becomes

$$R_{11} = \tfrac{1}{2}e^{f-h}(f'' + \tfrac{1}{2}f'^2 - \tfrac{1}{2}f'h' + \tfrac{1}{2}f'(q' + p'))$$
$$+\tfrac{1}{2}e^{f-q}(\ddot{f} + \tfrac{1}{2}\dot{f}^2 + \tfrac{1}{2}\dot{f}\dot{h} + \tfrac{1}{2}\dot{f}\dot{p}) = 0, \tag{25}$$

Assuming

$$f' + h' = 0 \tag{26}$$

and

$$\dot{f} + \dot{h} = 0. \tag{27}$$

We also find

$$\ddot{p} + \frac{1}{2}\dot{p}^2 = -rp'. \tag{28}$$

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Thus, (24) becomes

$$r(e^{-h})' + e^{-h} - 1 = 0. \tag{29}$$

The usual assumption is to consider that e^{-h} is independent of θ. In this particular case, from (29) we obtain the well known result

$$e^{-h} = (1 - \frac{r_s}{r}). \tag{30}$$

However, here we are interested in looking for more complete solution, in which e^{-h} is a function not only of r but also of θ. In searching for this possibility let us multiply (29) for $\sin\theta$. We have

$$r\sin\theta(e^{-h})' + \sin\theta e^{-h} - \sin\theta = 0. \tag{31}$$

This expression can also be written as

$$r\sin\theta(e^{-h})' + (a + r\sin\theta)'e^{-h} - (a + r\sin\theta)' = 0. \tag{32}$$

The two terms of (32) can be put together if we extend (32) in the form

$$(a + r\sin\theta)(e^{-h})' + (a + r\sin\theta)'e^{-h} - (a + r\sin\theta)' = 0. \tag{33}$$

Thus, (32) can be solved by writing

$$e^{-h} = (1 - \frac{\mathcal{A}(\theta)}{a + r\sin\theta}), \tag{34}$$

with $\mathcal{A}(\theta)$ an arbitrary function of θ. The prove that (34) is in fact a solution of (33) is straightforward. In fact by substituting (34) into (33) we get

$$(a + r\sin\theta)(1 - \frac{\mathcal{A}}{a + r\sin\theta})' + (a + r\sin\theta)'(1 - \frac{\mathcal{A}}{a + r\sin\theta}) - (a + r\sin\theta)' = 0 \tag{35}$$

Now it remains to determine $\mathcal{A}(\theta)$. We apply the well known procedure to derive the event horizon by setting

$$(1 - \frac{\mathcal{A}(\theta)}{a + r_s\sin\theta}) = 0, \tag{36}$$

with $r_s = const$, a fixed torus radius. So from (35) we get

$$\mathcal{A}(\theta) = a + r_s \sin\theta \tag{37}$$

and therefore (34) becomes

$$e^{-h} = (1 - \frac{a + r_s \sin\theta}{a + r\sin\theta}), \tag{38}$$

and since $e^f = e^{-h}$ we find that the line element can be written as

$$ds^2 = -(1 - \frac{a + r_s \sin\theta}{a + r\sin\theta})c^2dt^2 + \frac{dr^2}{(1 - \frac{a+r_s\sin\theta}{a+r\sin\theta})} + r^2d\theta^2 + (a + r\sin\theta)^2 d\phi^2. \tag{39}$$

This line element is reduced to the usual one when $a = 0$. In fact, when $a = 0$ we get

$$ds^2 = -(1 - \frac{r_s}{r})c^2dt^2 + \frac{dr^2}{(1 - \frac{r_s}{r})} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{40}$$

as expected.

IV. Final Remarks

The main goal of this work was to established a route to describe a black-hole solution for arbitrary genus g. For $g = 0$ we obtain the well known black-hole with S^2 as an event horizon. In this work, we have discovered how to derive a solution for $g = 1$, corresponding to the torus black-hole with event horizon topology $S^1 \times S^1$. It remains to generalize, for further work, our procedure to higher genus g. Moreover, it is interesting to observe that (39) is not singular at $r = 0$ as (40) but rather in $a + r\sin\theta = 0$. This means that there is singularity for $r \neq 0$. In fact, this result seems quite remarkable and perhaps can help to solve the old well known problem of the singularity at $r = 0$.

Another interesting observation is that our algorithm can also be used to find a kind a spiral black-hole solution. In fact from (21) we may also choose $A_q(\theta) = A_p(\theta) = 1$ and $B_q(\theta) = \alpha\theta$ and $B_p(\theta) = \alpha\theta\sin\theta$. This means that the last to terms of (39)

$$dl^2 \equiv r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{41}$$

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can be written in the alternative form

$$dl^2 = (r + \alpha\theta)^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{42}$$

When $r = 0$ the radius becomes

$$\rho \equiv \alpha\theta, \tag{43}$$

which correspond to the typical radius of a spiral curve. We are tempted to propose that (42) may be useful for describing galaxy dynamics, with a black-hole as a source system.

It remains to explore further the significance of the singularity at $r \neq 0$, for $g \neq 1$, as opposed to $r = 0$ for $g = 0$. In fact this result may provide an alternative solution of the long-standing problem of singularities in black-hole physics. It would be helpful to expand on this point by discussing its implications for the broader understanding of black hole singularities and potential avenues for further investigation.

For further research it may also be interesting to open new avenues to link our work for the existing literature on black-hole solution for varying topologies.

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